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# Mh4714 Week 4

# *Week 4*

### 0.0.1 Decimals, Sequences and the Rational Numbers.

- Recall that a rational number is by definition a ratio of two integers. The set $\mathbb{Q}$ of rational numbers is defined to be the set:

$$\left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}\right\}$$

It is clear that a finite decimal is a rational number: $0.d_1d_2 \dots d_n = \dfrac{d_1d_2 \dots d_n}{10^n}$.
An infinite recurring decimal is a Geometric series which we can now see also converges to a rational number:

$$\begin{gathered} 0.d_1d_2d_3 \dots d_nd_1d_2 \dots d_nd_1 \dots \\ = \left(\frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_n}{10^n}\right) + \left(\frac{d_1}{10^{n+1}} + \frac{d_2}{10^{n+2}} + \dots + \frac{d_n}{10^{2n}}\right) + \dots \\ = \frac{d_1d_2d_3 \dots d_n}{10^n} + \frac{d_1d_2d_3 \dots d_n}{10^{2n}} + \dots \end{gathered}$$

which is a geometric series converging to $\dfrac{\frac{d_1d2 \dots d_n}{10^n}}{1 - 10^n} = \dfrac{d_1d2 \dots d_n}{10^n - 1}$.

Example 0.1

The infinite decimal $0.45\dot{5}\dot{1}\dot{7}$ converges to $0.45 + \dfrac{1}{100}\dfrac{\frac{517}{1000}}{1 - \frac{1}{1000}} = \dfrac{45}{100} + \dfrac{517}{99900}$

We can call a decimal like this an *ultimately recurring* infinite decimal.

A finite decimal $0.d_1d_2\dots d_n$ can also be thought of as the ultimately recurring infinite decimal
$0.d_1d_2\dots d_n\dot{0}$

What we call *ultimately recurring* infinite decimals are frequently also called simply infinite recurring decimals.

If an infinite decimal $0.d_1d_2\dots$ converges to a number $r$ we simply write $0.d_1d_2\cdots = r$.

- Conversely, given any rational number, we can find an infinite decimal that converges to it.

We can use the "school" method of dividing a denominator into a numerator:

## Example 0.2

$$7 \,\big|\, \underline{1.\,{}_{1}0\,{}_{3}0\,{}_{2}0\,{}_{6}0\,{}_{4}0\,{}_{5}0\,{}_{1}0 \quad \cdots\cdots}$$
$$0.\,1\,4\,2\,8\,5\,7\,1 \quad \cdots\cdots$$

Looking at the process this way we can easily see that the decimal which we arrive at for any rational number $\frac{a}{b}, a, b \in \mathbb{N}$ must always be either finite or (ultimately) infinitely recurring because the only possible remainders on division by $b$ are $0, 1, 2, \dots b-1$. If we get remainder 0 at any stage then the process terminates and we get a finite decimal. Otherwise a remainder musr recur after at most $b-1$ divisions and once a remainder recurs then the digits in the decimal begin to recur.

Instead of using the 'school method', we can arrive at the same decimal expansion for a rational number in a more transparent manner as follows:

(i) We first determine how many $\frac{1}{10}$'s "fit into" $\frac{1}{7}$
Let $d_1$ be the largest integer such that

$$\frac{d_1}{10} \leq \frac{1}{7} \Rightarrow d_1 \leq \frac{10}{7} = 1\tfrac{3}{7} \Rightarrow d_1 = 1.$$

That is,

$$\frac{1}{7} = \left(1 + \frac{3}{7}\right) \times \frac{1}{10} = 1 \times \frac{1}{10} + \frac{3}{7} \times \frac{1}{10} = 1 \times \frac{1}{10} + \frac{3}{70}$$

(ii) We now determine how many $\frac{1}{10^2}$'s "fit into" the left over piece $\frac{3}{70}$.
Therefore let $d_2$ be the largest integer such that

$$\frac{d_2}{10^2} \leq \frac{3}{70} \Rightarrow d_1 \leq \frac{30}{7} = 4\tfrac{2}{7} \Rightarrow d_1 = 4.$$

That is,

$$\frac{1}{7} = \frac{1}{10} + \left(4 + \frac{2}{7}\right) \times \frac{1}{10^2} = \frac{1}{10} + \frac{4}{10^2} + \frac{2}{700}.$$

(iii) We now determine how many $\frac{1}{10^3}$'s "fit into" the left over piece $\frac{2}{700}$ and so on.

(iv) The 'left over piece' is getting arbitrarily small as we proceed and so we are constructing an infinite decimal which is converging to $\frac{1}{7}$.

Note that the above process is exactly parallel to the school method illustrated first.

- It is possible that two different decimals can converge to the same number.

## Example 0.3

$$0.1\dot{9} = 0.1 + 0.099999\cdots = 0.1 + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \dots$$

$$= 0.1 + \frac{\frac{9}{10^2}}{1 - \frac{1}{10}} = 0.1 + 0.1 = 0.2$$

But it is only by this phenomenon of repeating 9's that we can get two different decimals converging to the same number.
If we ban infinitely repeating 9's then every rational number has a unique decimal representation.

*0.0.1.1 Real numbers v. Rational numbers.* We see that every finite and every infinite recurring decimal is a rational number and every rational number equals either a finite or infinite recurring decimal.

It follows therefore that an infinite non-recurring decimal does not converge to a rational number.

## Example 0.4

The infinite non-recurring decimal

$$\sum_{k=1}^{\infty} \frac{1}{10^{\frac{k(k+1)}{2}}} = 0.101001000100001\ldots$$

does not converge to a rational number.

The irrational numbers are therefore called into existence by adding one more axiom to the axioms defining the rational numbers:

**Completeness Axiom for Real Numbers**: Every infinite decimal converges to a real number.

## Example 0.5

The Completeness example guarantees that there is a real number which is the limit of the decimal

$$0.101001000100001\ldots$$

That is, there is some real number, call it $r$ such that

$$r = 0.101001000100001\ldots$$